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Multiple scattering in arrays of axisymmetric wave-energy devices. Part 1. A matrix method using a plane-wave approximation

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A technique is developed to model the multiple scattering of surface waves in an array of axisymmetric wave-energy devices. The matrix equation which results is inverted to yield the exciting forces, the added-damping and added-mass matrices, the optimal power absorption and the optimal device responses. The matrix method is also used on the previously unstudied problem of an unconstrained array. Finite-size effects of devices are shown to be important in producing phase-shifts, which shift the unconstrained frequency response, but leave the optimal energy absorption virtually unchanged.

1. Introduction

Recently much effort in the field of wave power has been concentrated on the problem of the interactions between wave-energy devices in arrays. This interest is due partly to the fact that a practical wave-power station situated off the British coast might well consist of many individual devices uniformly spaced in a linear array. Partly, also, this research is driven by the knowledge that, when many bodies are placed together in a configuration, the power absorption per body may be substantially different from that of the isolated units, due to hydrodynamic coupling between the bodies. Early work by Budal (1977) demonstrated, by making many simplifying assumptions, that considerable enhancement of performance could result in linear arrays at suitable spacing-to-wavelength ratio. More recently, Evans (1980) improved on the previous work and produced a full optimisation which showed that the interactions between array elements could be still more favourable. Both these theories are of general validity, but calculations from them were only performed for the case when the devices are 'point absorbers', so small that their scattering effect is negligible. This is because of the difficulty, in the general case, of determining the functions denoted by Evans as $f_n(\theta)$, by Budal as $b_{lm}(\theta)$, and here as $\gamma_m(\theta)$.

An analysis that accounts properly for scattering is therefore needed. Also, both Evans and Budal were seeking optimum results, which implies that devices are linked operationally via some controller of phase and amplitude. This may well be impractical, and it is therefore necessary to know how amplitudes and phases of device motions adjust themselves, and thus what the true (non-optimal) energy absorption is.

More recently Thomas & Evans (1981) have used the point-absorber theory to study arrays of heaving thin ships and semi-immersed spheres, with particular attention to the optimal body-displacement amplitudes. It was found that the amplitudes for such devices were in general much larger than the wave height. Also, the maximum power was significantly reduced when the body amplitudes were constrained. It is therefore important to examine the optimal amplitudes as well as the optimal power; results presented in § 7 for another device geometry show that the body amplitudes are not always excessive.

The main effort of the present work is to take into account the scattering of waves between devices in an array, and this is also the goal of the work done by Count & Jefferys (1980) and Greenhow (1980). The former authors undertook a numerical study of the full hydrodynamic interaction between heaving axisymmetric buoys, using a boundary-element technique. This method is, in principle, more accurate than the present work, but becomes computationally prohibitive if more than a few devices are considered. Conversely, Greenhow (1980) sought to investigate diffraction effects between N semi-immersed spheres in an array by considering $N-1$ nearest-neighbour interactions; each of these is a two-body problem, in which only first- and second-order scattering are allowed.

These last two papers illustrate certain ways in which the introduction of scattering can modify the properties of an array, and one task of the present work is to try to verify their conclusions, and to gain an understanding of the reason for these modifications. The prototype problem considered here and in both these papers is that of two heaving spheres, with results presented for the damping coefficients and q-factor. Both papers show that the curve of the off-diagonal damping coefficient is shifted towards higher separation (this has to be deduced from figure 15 of Greenhow's paper), but they disagree about the effect on the q-factor, when scattering is included, Count & Jefferys maintaining there is virtually no change. It is impossible to resolve this contradiction without a method that allows some insight into the interaction, and one object of the present work was to aim for a method simple enough that insight was possible. It will be seen that some understanding has been achieved in certain aspects of the array problem when scattering is present, but there is still much that is not clear, and the hope is that further work will elucidate this.

The method described here to investigate the actual performance of arrays, including *multiple* scattering, is a fairly flexible one. In common with all the other array studies mentioned it is based on linearized water-wave theory; unlike other investigations it achieves substantial simplification by using a 'plane-wave' approximation which allows the multiple-scattering problem to be posed as a matrix equation. In this paper it is limited to uniformly spaced linear arrays of identical axisymmetric devices making heaving motions; this is purely for ease of application. It would be straightforward to consider unequal spacings, and it would also be possible to model in this way arrays that are not linear, although this would be more difficult if the number of devices was not small. (Triangular and rectangular arrays would prove interesting examples for study using this method.) The simplicity of the method relies largely on the restriction to axisymmetric bodies, and the heaving motions thereof, so a more complex technique is necessary to model more realistic devices.

Work by Ohkusu (1973, 1974) also deals with the interactions of axisymmetric bodies, including multiple scattering. The method described in those papers is in principle more accurate than the one considered here (in that the plane-wave approximation is not needed) but is computationally much less convenient when there are more than a few devices. Ohkusu applies the method to fixed vertical circular cylinders, but there is no reason preventing an extension to arbitrary axisymmetric geometries also absorbing energy. It will make an interesting comparison to study, with the two different methods, an array of, say, five devices absorbing energy. For the present, the investigation will use the plane-wave approximation, which results in a simple, compact matrix-equation formulation.

In § 2 the approximations used in setting up the matrix equation are outlined, and then the general problem of an array absorbing energy is solved in §§ 3, 4, the former being a simple illustration in which the array consists of only two bodies. (This is still a very complex problem for the full hydrodynamic interaction.) The added-damping and added-mass matrices for the array are then derived in § 5. The details of the particular device investigated, the submerged cylindrical duct of Simon (1981), are given in § 6, and the numerical results are calculated and presented in § 7.

Part 2 of this paper, currently in preparation, will examine another technique for studying array interactions. This technique avoids the plane-wave approximation and allows the energy-extraction mode to be horizontal (for example a 'Lancaster flounder' or a submerged sphere constrained to move in sway and surge). The case of an infinite array is also studied, and results presented for other axisymmetric device geometries.

2. The approximations involved

Consider the effect of one wave-energy device (device 1), on another device (device 2), due to the former diffracting, and radiating, waves. (Note that the radiation need not be present, since the bodies could be fixed, and diffraction could be negligible in the 'point absorber' limit. Thus these two effects are independent, and there will later be cause for omitting one or other.) Both effects contribute to the total potential in two ways; at large distance from device 1 there are extra waves radiating outwards, but close to device 1 there is also a *local* wave field which decays with distance. However, only the potential in the vicinity of device 2 matters for the interaction. In what follows, this potential will be approximated by that of a *plane wave* of an appropriately chosen amplitude. There are two distinct approximations involved here; one involves the devices being spaced wide enough apart so that the local wave field has no influence, and this has been shown to be valid, even at small spacing, in two dimensions (see Srokosz & Evans 1979). There is good reason to believe it is equally applicable in three-dimensional problems. The outer approximation models a diverging wave as plane; although this is clearly rather crude, it does preserve some characteristics of the wave, as the following argument shows.

Take axes (r_i, θ_i, z) centred on the ith device, with $\theta_i = 0$ normal to the line of centres $(i = 1, 2)$, as in figure 1.

Then a general wave potential emanating from device 1 can be written

$$\phi = e^{-Kz} \sum_{n=-\infty}^{\infty} (-i)^n a_n H_n^{(2)}(Kr_1)\, e^{in\theta_1} \tag{1}$$

(with $\mathscr{R}[\dots e^{i\omega t}]$ understood). This can be expressed in the form

$$e^{-Kz} \sum_{n=-\infty}^{\infty} a_n \left[\sum_{m=-\infty}^{\infty} J_m(Kr_2)\, H_{n-m}^{(2)}(KS)\, e^{im\theta_2}(-i)^m \right], \tag{2}$$

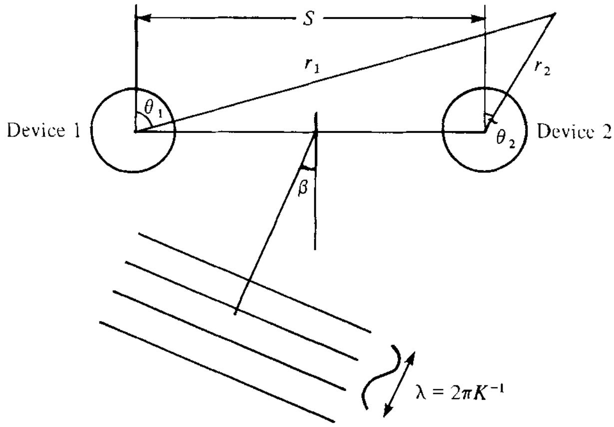

FIGURE 1. The incident wave, the devices and the local co-ordinates.

using Graf's addition theorem (Abramowitz & Stegun 1964), or

$$e^{-Kz} \sum_{m=-\infty}^{\infty} \left[\sum_{n=-\infty}^{\infty} a_n H_{n-m}^{(2)}(KS) \right] J_m(Kr_2)\, e^{im\theta_2}(-i)^m, \tag{3}$$

assuming the order of summation is not important. The term in this potential that is independent of θ_2 is equal to

$$e^{-Kz} \left[\sum_{n=-\infty}^{\infty} a_n H_n^{(2)}(KS) \right] J_0(Kr). \tag{4}$$

If this is to be replaced by a *plane* wave potential near device 2, moving away from device 1 and of amplitude A, then (3) would become

$$A\, e^{-Kz} \sum_{m=-\infty}^{\infty} J_m(Kr_2)\, e^{im\theta_2}(-i)^m, \tag{5}$$

which has the same azimuthally independent component as (4) if

$$A = \sum_{n=-\infty}^{\infty} a_n H_n^{(2)}(KS), \tag{6}$$

which is the amplitude of the wave potential (1) *measured at the centre of device* 2. As with the single duct of Simon (1981), it is the component (4) which determines the exciting force for any *axisymmetric* body in heave. Also, for diffraction by an axisymmetric body this will form the *dominant* contribution except at particular frequencies.

Thus the plane-wave approximation with the 'obvious' choice of amplitude seems justifiable provided the bodies are axisymmetric; with this restriction it is possible to pose a very hard multiple-scattering problem in the form of a simple matrix equation. The errors introduced by this approximation are considered in the appendix.

3. The two-body problem

Before describing the interactions between devices in an array, it is useful to describe the coupling between an isolated single device of the type being considered and the waves. This coupling can be characterized by three complex constants $\hat{D}$, $\hat{E}$, $\hat{F}$ as

FIGURE 2. The devices, the incident wave represented by I_1 and I_2, and the interaction effects represented by c and d.

follows. Suppose that the device is making vertical motions of amplitude ξ, then define $\hat{D}$ such that the far-field potential takes the form

$$\omega L^2 K \xi \hat{D} e^{-Kz} H_0^{(2)}(Kr), \tag{7}$$

which corresponds to outgoing circular waves of amplitude

$$\hat{D}(KL)^2 \xi |H_0^{(2)}(Kr)|. \tag{8}$$

Thus $\hat{D}$ is dimensionless, L being a typical radius of the device.

Conversely, suppose a plane wave of amplitude $\mathscr{A}$ is impinging on the fixed device, then the exciting force is given by

$$\text{force} = \pi \rho g L^2 \mathscr{A} \hat{E}, \tag{9}$$

which defines the dimensionless constant $\hat{E}$. Both $\hat{D}$ and $\hat{E}$ depend simply on the geometry of the device positioned relative to the free surface, and the frequency; as seen later they are related to one another. However, if the device is allowed to move in response to the forcing (9), and possibly extract energy, then the response amplitude will be given by another relation

$$\xi = \hat{F} \mathscr{A}. \tag{10}$$

The dimensionless constant $\hat{F}$ involves not only the forcing, but also the equation of motion (in its frequency-domain form) of the device.

Now the interactions of devices in an array can be studied; the hydrodynamic behaviour of the array depends only on the hydrodynamic properties of individual devices, and these properties are embodied in $\hat{D}$, $\hat{E}$ and $\hat{F}$. It will be assumed that the devices in an array are identical (for the sake of convenience), and so these constants apply to each individual device.

As a simple illustration of the method, consider the case when the array consists of only two bodies. There will be no difficulty in extending this to more bodies in § 4. Consider an incident wave of unit amplitude impinging on the two-body array, with wave-crests making an angle β with the line of the array, as in figure 1. For convenience set the phase of the incident wave to be zero in between the devices; then this wave can be expressed as

$$\left.\begin{aligned} \Phi^{\mathrm{I}} &= g\omega^{-1} I_1 \exp(-Kz + iKr_1 \cos(\theta_1 - \beta)) \\ &= g\omega^{-1} I_2 \exp(-Kz + iKr_2 \cos(\theta_2 - \beta)), \end{aligned}\right\} \tag{11a}$$

with the phase factors I_1, I_2 given by

$$I_1^* = I_2 = \exp(\tfrac{1}{2} iKS \sin \beta). \tag{11b}$$

From the results of § 2 it is possible to model all the interaction between the two bodies by incorporating plane waves of complex amplitudes c, d as shown in figure 2.

The plane wave d will be due to all the scattering (of I_1 and c), and/or radiation, from device 1, and c will be due to device 2. Now an axisymmetric device will scatter a plane wave

$$\sum_{n=0}^{\infty} (-i)^n \epsilon_n J_n(Kr) \cos(n\theta - n\chi) \tag{12a}$$

in such a way as to add a diffraction potential

$$\sum_{n=0}^{\infty} (-i)^n \epsilon_n A_n H_n^{(2)}(Kr) \cos(n\theta - n\chi). \tag{12b}$$

Here the scattering coefficients A_n are complex constants independent of χ. It is this potential (12b) which must be added to an existing wave-field to account for scattering. Denoting $H_n^{(2)}(KS)$ by H_n, we have

$$d = I_1\left[\sum_0^{\infty} \epsilon_n(-i)^n A_n H_n \cos(\tfrac{1}{2}n\pi - n\beta)\right] + c\left[\sum_0^{\infty} \epsilon_n(-i)^n A_n H_n \cos n\pi\right] + (KL)^2 \hat{D}\xi_1 H_0, \tag{13a}$$

$$c = I_2\left[\sum_0^{\infty} \epsilon_n(-i)^n A_n H_n \cos(\tfrac{1}{2}n\pi + n\beta)\right] + d\left[\sum_0^{\infty} \epsilon_n(-i)^n A_n H_n \cos n\pi\right] + (KL)^2 \hat{D}\xi_2 H_0. \tag{13b}$$

Here ξ_1, ξ_2 are the amplitudes of device oscillations. The three terms that contribute to d are due respectively to the scattering of I_1 and c off device 1, and the radiation resulting from that device oscillating. All these effects give rise to a potential at device 2 which is replaced, using the approximations of § 2, by a plane-wave potential which has the same amplitude as the true potential has at the centre of device 2. This amplitude is what is written on the right-hand side of (13a). The terms related to scattering are simply (12b) with $r = S$ and appropriate values of θ and χ.

Now device 1 only 'detects' the other device via the plane wave c, so the response of the former is as if it were *isolated* with incident waves I_1 and c from the appropriate directions. Due to *axisymmetry* these directions are irrelevant and ξ_1, ξ_2 are therefore determined as though each device were isolated, with incident waves $I_1 + c$, $I_2 + d$ respectively. So

$$\xi_1 = \hat{F}(I_1 + c), \quad \xi_2 = \hat{F}(I_2 + d). \tag{14}$$

Writing
$$M = \hat{D}\hat{F}(KL)^2, \quad \hat{A}_n = \epsilon_n(-i)^n A_n + M\delta_{n0}, \tag{15}$$

allows (13) and (14) to be combined as

$$d = I_1 P^{(1)} + cP^{(3)}, \quad c = I_2 P^{(2)} + dP^{(3)}, \tag{16a}$$

where
$$P^{(j)} = \sum_{n=0}^{\infty} \hat{A}_n H_n^{(2)}(KS) \cos(n\gamma^{(j)}), \tag{16b}$$

with
$$\gamma^{(j)} = \tfrac{1}{2}\pi - \beta, \tfrac{1}{2}\pi + \beta, \pi, 0 \quad (j = 1, 2, 3, 4). \tag{16c}$$

($\gamma^{(4)}$ will be used later.)

Thus c and d can be determined, and from these the exciting forces which are proportional to $I_1 + c$ and $I_2 + d$, which follows from the argument preceding (14). It is then clear that the energy absorptions of the two devices in the array have been multiplied (due to the interaction) by factors

$$|I_1 + c|^2, \quad |I_2 + d|^2 \tag{17}$$

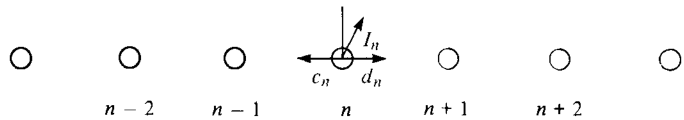

FIGURE 3. The three plane waves I_n, c_n and d_n which scatter off, and force the motions of, the nth device.

respectively. This enables the energy absorption for the (two-body) array to be evaluated as

$$2\bar{q}P_1, \tag{18}$$

where P_1 is the energy absorption of a single device in isolation, and $\bar{q}$ is an 'averaged interaction factor':

$$\bar{q} = \tfrac{1}{2}\{|I_1+c|^2+|I_2+d|^2\}. \tag{19}$$

The devices are aiding one another owing to their mutual positioning if $\bar{q} > 1$, but are hindering each other if $\bar{q} < 1$. It is expected that there will be certain combinations of spacing and angle of incidence for which there is a favourable interaction, and other combinations where there will be (unavoidable) reduction in performance.

Notice that this method also gives the true energy absorption of a 'point-absorber' array, simply by setting $A_n = 0$ for all n; conversely the problem of diffraction off an array of fixed bodies is achieved by setting $\hat{F} = 0$, and this proves useful for evaluating *optimal* power absorption later. Further notice that the extension to a general array of N bodies is straightforward, achieved simply by introducing more pairs of plane waves (c, d), with a total of $N(N-1)$ quantities to be determined. For a linear array, however, this reduces to $2(N-1)$ owing to some of these introduced waves amalgamating.

4. The N-body, linear array

Consider the case of uniform spacing. Put the new phase factors

$$I_n = \exp\{-iKS(n-\tfrac{1}{2}(N+1))\sin\beta\} \quad (1 \leqslant n \leqslant N), \tag{20}$$

so that the phase is again set to zero in the centre of the array, as in (11). In this case the plane wave of complex amplitude c_n is due to *all* devices numbered higher than n, and similarly d_n is the complex amplitude of the plane wave due to all lower numbered devices, as shown in figure 3. Thus c_n is a sum of the diffraction radiation effects from other devices. There is an interesting point here, for although a contribution to c_n from device $n+2$, say, will be *modified* by device $n+1$, this modification will be properly accounted for in the sum as part of the contribution from device $n+1$. The matrix equation is thus

$$d_n - \sum_{1\leqslant m<n} \{c_m P^{(3)}_{mn} + d_m P^{(4)}_{mn}\} = \sum_{1\leqslant m<n} I_m P^{(1)}_{mn}, \tag{21a}$$

$$c_n - \sum_{n<m\leqslant N} \{d_m P^{(3)}_{mn} + c_m P^{(4)}_{mn}\} = \sum_{n<m\leqslant N} I_m P^{(2)}_{mn}, \tag{21b}$$

$$d_1 = c_N = 0. \tag{21c}$$

Here $$P^{(j)}_{mn} = P^{(j)}_{nm} = \sum_{r=0}^{\infty} \hat{A}_r H^{(2)}_r(KS|n-m|)\cos(r\gamma^{(j)}). \tag{22}$$

$P^{(4)}_{mn}$ is due to waves that are travelling along the array scattering 'ahead', whereas $P^{(3)}_{nm}$ is due to scattering 'behind' the device; the angles $\gamma^{(j)}$ are as in (16).

As with (11), it is possible to explain (21) as follows. Every I_m, c_m or d_m scattering off device m will contribute to the total potential at device n ($n \neq m$). The contributions from all devices are replaced by plane-wave potentials of the correct amplitudes, with the direction of propagation chosen according to the sign of $n-m$. These plane-wave potentials add owing to linearity, to form c_n and d_n. Using (15) it is again possible to include the effects of devices that are radiating owing to wave-forces on them.

In practice the infinite sum (22) can be truncated after a few terms because of the rapid decay of the scattering coefficients A_n, so each matrix coefficient $P^{(j)}_{mn}$ requires few Hankel-function evaluations. For an array with non-uniform spacing there are $2N(N-1)$ values $P^{(j)}_{mn}$, but this reduces to $4(N-1)$ when the spacings are equal. (In fact, $P^{(4)}_{1N}$ need not be computed because of (21c).)

One very convenient feature of this matrix form of the scattering problem, is that the matrix that multiplies the $2(N-1)$ unknowns c_n and d_n involves $P^{(3)}_{mn}$ and $P^{(4)}_{mn}$ only, and is thus *independent of* β, the angle of incidence. This means that, for a given value of KS, only one call to a standard NAG computer routine enables the matrix equation to be inverted for many different 'right-hand sides', corresponding to many different values of β. This significantly reduces the computation time, although at the cost of increased storage.

After inversion, the averaged interaction factor is given by

$$\frac{P_{\text{abs}}}{NP_1} = \bar{q} = \frac{1}{N}\sum_{n=1}^{N} |I_n + c_n + d_n|^2, \tag{23}$$

where P_{abs} is the energy absorbed by the array.

The numerical results obtained later show that $\bar{q}$ can indeed be significantly different from unity, and can be very sensitive to changes of wavelength and angle of incidence when N is large.

5. The diffraction and radiation problems

By controlling the amplitudes and relative phases of the device responses, the energy absorption of the array can be varied. There will be a unique maximum value of the energy it is possible to extract with a given array; this optimal absorption will occur only if the phases are in the correct relationship, which is determined by all the many parameters involved. However it is shown in Evans (1980) that, in order to *evaluate* the optimal energy extraction of an array at a given angle of incidence, it is sufficient to know

(a) the exciting forces F_n ($n = 1, \ldots, N$) on the bodies in the array when they are held fixed in waves at the given angle, and

(b) the added-damping matrix. The disturbance caused by the motion ξ_j of the jth body will produce a contribution to the force on the ith body. This contribution is written

$$-(\bar{B}_{ij}\dot{\xi}_j + \bar{M}_{ij}\ddot{\xi}_j), \tag{24}$$

arbitrarily splitting it into components in phase with velocity and acceleration respectively. The real symmetric matrices $\bar{\mathbf{B}}$ and $\bar{\mathbf{M}}$ are the added-damping and added-mass matrices, but the latter will not contribute when the time-averaged power absorption is sought. Note that the exciting forces, added-damping and added-mass coefficients are all frequency-dependent.

Although the quantities $\bar{B}_{ij}$ are properties of the radiation problem, where one body is forced to oscillate with all other bodies held fixed, the reciprocity between radiation and diffraction allows the matrix elements to be computed from the exciting forces F_n known at all angles of incidence. (In practice, using the symmetry of the configuration as in (26) and (27), this meant about 50 angles in $[0, \frac{1}{2}\pi]$.) The reciprocity, which is widely applicable in wave problems, is proved for the present case in Srokosz (1979*a*). It is derived by applications of Green's theorem and the method of stationary phase.

5.1. *The diffraction problem*

The exciting forces F_n are obtained by solving the matrix equation for fixed bodies ($\hat{F} = 0$). Writing

$$\gamma_n(\beta) = (I_n + c_n + d_n)_{\hat{F}=0} \quad \text{gives} \quad F_n(\beta) = \pi\rho g L^2 \hat{E}\gamma_n(\beta). \tag{25}$$

The complex quantity $\gamma_n(\beta)$ represents the amount by which the exciting force F_n has been *augmented* owing to array effects. The reciprocity between radiation and diffraction allows the dimensionless form of the damping matrix to be evaluated as

$$\begin{aligned} B_{nm} &= (2\pi)^{-1}\,|\hat{E}|^2 \int_0^{2\pi} \gamma_n(\theta)\,\gamma_m^*(\theta)\,d\theta \\ &= \pi^{-1}|\hat{E}|^2 \int_0^{\frac{1}{2}\pi} \{\gamma_n(\theta)\,\gamma_m^*(\theta) + \gamma_{N+1-n}(\theta)\,\gamma_{N+1-m}^*(\theta)\}\,d\theta \end{aligned} \tag{26}$$

because
$$\gamma_n(\pi-\theta) = \gamma_n(\theta) \quad \text{and} \quad \gamma_n(2\pi-\theta) = \gamma_{N+1-n}(\theta). \tag{27}$$

The last relation is a consequence of the symmetry of the array about its midpoint due to uniform spacing, and the choice (20). Note that

$$\gamma_n(2\pi-\theta) = \gamma_n^*(\theta) \tag{28}$$

is no longer true when scattering is included. This is in contrast, therefore, to the point-absorber case, and in particular to the discussion around equation (4.12) of Evans (1980).

Note that $|\hat{E}|^2$ is simply the damping coefficient for a single device, as seen by considering (26) with $N = 1$, in which case $\gamma_1(\theta) = 1$ for all θ.

The q-factor (which measure the enhancement of performance due to interactions) and the optimal power are then given by

$$Nq_{\text{opt}}(\beta) = K\frac{P_{\text{opt}}}{P_{\text{inc}}} = |\hat{E}|^2 \sum_{i,j} \gamma_i(\beta)\,(\mathbf{B}^{-1})_{ij}\,\gamma_j^*(\beta). \tag{29}$$

Equations corresponding to (26) and (29) were first derived by Evans (1980), but calculations from it were performed for the point absorber case only. Notice that q_{opt} is a function of N, β, KS, and of the scattering coefficients A_n. It does not depend on any other property of the single-body problem, and this includes the energy-extraction process. However, the displacements ξ_n necessary to achieve this optimum extraction depend on $\hat{E}$, and the remaining properties of the single-body problem must be such that these optimum displacements are achieved.

By (23) and (29), it is clear that

$$\frac{P_{\text{abs}}}{P_{\text{opt}}} = \frac{\bar{q}KP_1}{q_{\text{opt}}P_{\text{inc}}} \leqslant 1. \tag{30}$$

The numerical values calculated later for an array of identical devices of a specific type suggest that the true energy absorption will be significantly less than optimum (since the ratio (30) is typically 0·6–0·8) when the number of devices in the array is large. There is thus scope for further optimisation, but it is by no means clear how this is to be achieved, bearing in mind that the devices in practical configurations would be identical. (The optimum energy capture for a *finite* array requires different damping and spring/inertia constants for each device. This is shown in Srokosz (1979*b*, p. 35). This corresponds to different values of $\hat{F}$ for each device, but this is not at variance with the present method, since $\hat{F}$ does not affect the values of $\gamma_j(\theta)$ or B_{ij}.)

5.2. *The radiation problem*

The more straightforward way to derive the added-damping matrix is by considering the radiation problems where one body oscillates in the presence of $N-1$ fixed bodies; this way also yields the added-mass matrix. There are N such problems, but symmetry allows the number to be reduced to $\frac{1}{2}N$ (N even), or $\frac{1}{2}(N+1)$ (N odd). It is useful to derive two relations between quantities involved in the single-body problem before writing down the matrix equation which represents the radiation problem.

Let ϕ_{R} and ϕ_{S} be respectively the radiation and scattering potentials for an isolated (axisymmetric) body; also take $\mathscr{S}$ to be the closed surface formed by the wetted surface of the body, the free-surface, a cylindrical closure of large radius, and the fluid bottom at $z=\infty$. Green's theorem, applied inside $\mathscr{S}$, with ϕ_{R} and ϕ_{S} gives

$$\hat{D} = \tfrac{1}{2}i\pi\hat{E}; \tag{31a}$$

also, using this theorem with $\phi_{\text{R}}-\phi_{\text{R}}^*$ and ϕ_{S}, as in Davis (1976), gives

$$\hat{D} = (2A_0+1)\hat{D}^*. \tag{31b}$$

Thus

$$-\frac{\hat{E}}{\hat{E}^*} = \frac{\hat{D}}{\hat{D}^*} = 2A_0+1 = e^{-2i\alpha}, \quad \text{say;} \tag{31c}$$

this definition of the phase angle α is the same as is used in equation (48) of Simon (1981) for the particular case of the cylindrical duct, with A_0 written there as A.

The matrix equation for the radiation problem can now be written as (21) and (22) with $\hat{F}=0$ (i.e. $\hat{A}_0 = A_0$) and with right-hand sides

$$\left.\begin{matrix} H_0^{(2)}(KS|n-q|) & (n>q) \\ 0 & (n \leqslant q) \end{matrix}\right\} \quad \text{for} \quad d_n, \tag{32a}$$

$$\left.\begin{matrix} 0 & (n \geqslant q) \\ H_0^{(2)}(KS|n-q|) & (n<q) \end{matrix}\right\} \quad \text{for} \quad c_n. \tag{32b}$$

Here device q is oscillating, with amplitude $\xi_q = (K^2L^2\hat{D})^{-1}$; denote the solution by $c_n^{(q)}, d_n^{(q)}$ $(1 \leqslant n \leqslant N)$. Then, by definition of the added-mass and damping matrices with the appropriate non-dimensionalization, we have

$$\begin{aligned} \pi\rho g L^2 \hat{E}(c_p^{(q)} + d_p^{(q)}) &= -(i\omega \bar{B}_{pq} - \omega^2 \bar{M}_{pq})\,\xi_q \\ &= -\omega^2(iB_{pq} - M_{pq})\,\tfrac{1}{2}\pi^2 L^4 \rho K \xi_q \\ &= -(iB_{pq} - M_{pq})\,\tfrac{1}{2}\pi^2 L^2 \rho g/\hat{D}. \end{aligned} \tag{33}$$

Using (31) now gives

$$B_{pq} + iM_{pq} = \frac{2i}{\pi}\hat{D}\hat{E}(c_p^{(q)} + d_p^{(q)}) = -\hat{E}^2(c_p^{(q)} + d_p^{(q)}) = |\hat{E}|^2\, e^{-2i\alpha}(c_p^{(q)} + d_p^{(q)}). \tag{34}$$

Adding the force on the qth body due to its own oscillation in isolation gives

$$D_{\mathrm{r}}^{-1} B_{pq} = \delta_{pq} + \mathscr{R}\{e^{-2i\alpha}(c_p^{(q)} + d_p^{(q)})\}, \tag{35a}$$

$$M_{pq} = M^{(1)}\delta_{pq} + D_{\mathrm{r}}\mathscr{I}\{e^{-2i\alpha}(c_p^{(q)} + d_p^{(q)})\}. \tag{35b}$$

In these relations $D_{\mathrm{r}} = |\hat{E}|^2$ is the added-damping coefficient, and $M^{(1)}$ the added-mass coefficient, for an isolated device; δ_{pq} is the Kronecker delta.

For the particular device geometry considered in this paper it was found that calculations of B_{nm} using (26) and (35*a*) were in close agreement. However, certain shortcomings were observed, possibly due to inaccuracies introduced by the plane-wave approximation. For the integral (26) was not necessarily real *numerically*, and the matrices given by (35) were not necessarily symmetric. It was felt that the results were still meaningful despite this, since the imaginary part of (26), the antisymmetric part of (35*a*), and the disagreement between the two methods were all typically less than 10^{-2}, and never more than 3 % in any of the specific cases computed.

6. An application to a particular device

In this section, the foregoing method is applied to an array of cylindrical resonant ducts, using the results detailed in Simon (1981). We may note that another simple geometry available for study is that of the submerged sphere, and Srokosz (1979*b*) presents comprehensive results for the sway and heave added-masses and dampings of such a device. Once the scattering coefficients for the sphere have been derived, (by a straightforward extension of the analysis of that paper), the matrix method can be applied to an array of spheres constrained to move in heave only. This work is under progress, and will appear in part 2.

The scattering coefficients for the cylindrical duct were computed for $n = 1, \ldots, 5$, as well as for $n = 0$ which is given by equations (48*a*–*c*) of Simon (1981). This is achieved by replacing the quantity ζ in (37), the function $\bar{H}(x)$, and the function $\bar{\psi}(z)$ in (33) of that paper by

$$\zeta_n = \frac{C_n}{iA_n}\mu\{J_n'(\mu)\}^2 = \mu\{J_n'(\mu)\}^2\left\{\frac{1+A_n}{iA_n} - \frac{Y_n'(\mu)}{J_n'(\mu)}\right\} \geqslant 0, \tag{36}$$

$$\bar{H}_n(x) = -2xI_n'(x)\,K_n'(x) \geqslant 0, \tag{37}$$

$$\bar{\psi}_n(z) = (z^2-1)^{-\frac{1}{2}} + \gamma z^{-1} + \beta z^{-2} \quad \text{respectively.} \tag{38}$$

It was found that typically $A_5/A_0 \ll 10^{-4}$, and so the infinite sum (22) for $P_{nm}^{(j)}$ was replaced by $\Sigma_{r=0}^{4} \cdots$.

For the duct, the flow Q_n takes the place of the amplitude ξ_n of device oscillations. The quantity M of (13) is given by

$$M = -K_A^2 e^{-2\tau} \bigg/ \left\{ D_r + D_e + \frac{2i}{\pi\mu}\left(\frac{Z}{a} + \hat{l} - \frac{1}{\mu}\right)\right\} \tag{39a}$$

$$K_A^2 e^{-2\tau} = D_r e^{-2i\alpha} = D_r(1+2A_0), \tag{39b}$$

in the notation of Simon (1981). All the computations were performed with $\sigma = h/a = \tau/\mu = 0{\cdot}5$; and where M was used in the study of the energy absorption of an unconstrained array, Z/a and D_e were determined by conditions (97) of that paper with $\mu_0 = 1$. Note that a distinction must be understood between the 'zero-scattering' limit and the 'point-absorber' limit. The latter is the limit as $\mu = Ka \to 0$, which gives $A_n \to 0$ for all n, $D_r \to 1$, $K_A \to 1$ and $\hat{l} \to \hat{l}_0 > 0$. The former limit, which is interesting because it separates scattering from other effects of finite size, is achieved by setting $A_n = 0$ for all n, with all other properties of the device fixed, including α. Thus (33) was used first to evaluate α, and then A_0 set to zero, when considering this limit.

At this point it is worth observing that, if the geometry of the duct has been optimised, M remains close to $-\frac{1}{2}e^{-2i\alpha}$ over a broad band around the tuning frequency μ_0. Hence $\hat{A}_0 = A_0 + M$ remains close to $-\frac{1}{2}$ over this band, whereas this would be the same as the value of M in the 'zero-scattering' limit. It is possible that this is important in unconstrained arrays, since $\hat{A}_0$ constitutes the dominant effect for practical device sizes.

7. Results and discussion

Figure 4(*a*) shows contours of the averaged interaction factor $\bar{q}$, for an array of five devices, plotted against dimensionless spacing and angle of incidence. Each device has a diameter of one third of the spacing S. Of note are the very favourable interactions for $\beta = 0$ and $KS \approx 4$, and the unfavourable interaction $\beta = 0$, $KS \approx 7$ and $\beta = \frac{1}{2}\pi$, $KS \approx 3$. For the sake of comparison the zero scattering limit is also shown, in figure 4(*b*). It is clear that the introduction of scattering tends to 'shift' the pattern of response towards higher frequencies, as well as modifying this pattern. Thus it does seem that scattering is important in arrays.

Figure 4(*c*) shows the contours of $\bar{q}$ for 9 devices. Again $a/S = \frac{1}{6}$. It is clear that increasing the number of devices accentuates the favourable and unfavourable interactions, and makes the transition between the two sharper. Investigation of these contours for a large number of devices shows that these transition lines are given by $KS(1 \pm \sin\beta)/2\pi$ = an integer; the case of an infinite array links these lines to the divergence of an infinite series, or alternatively a contour integral. This will be shown in part 2 of this work, where computations are presented for the infinite-array case.

Figure 5 shows how the energy absorption is distributed throughout an array of nine devices, both with and without scattering, in beam and head seas. The overall form of this distribution is unchanged by including scattering, but the average level is modified. The figure shows also that the term 'attenuator' is strangely apt for head seas, with the rear devices contributing virtually no power. This suggests that it is a mistake to design attenuator devices with a large number of active elements, as noted by other authors, including Count & Jefferys (1980).

Before moving on to consider the optimal energy absorption of the array, it is

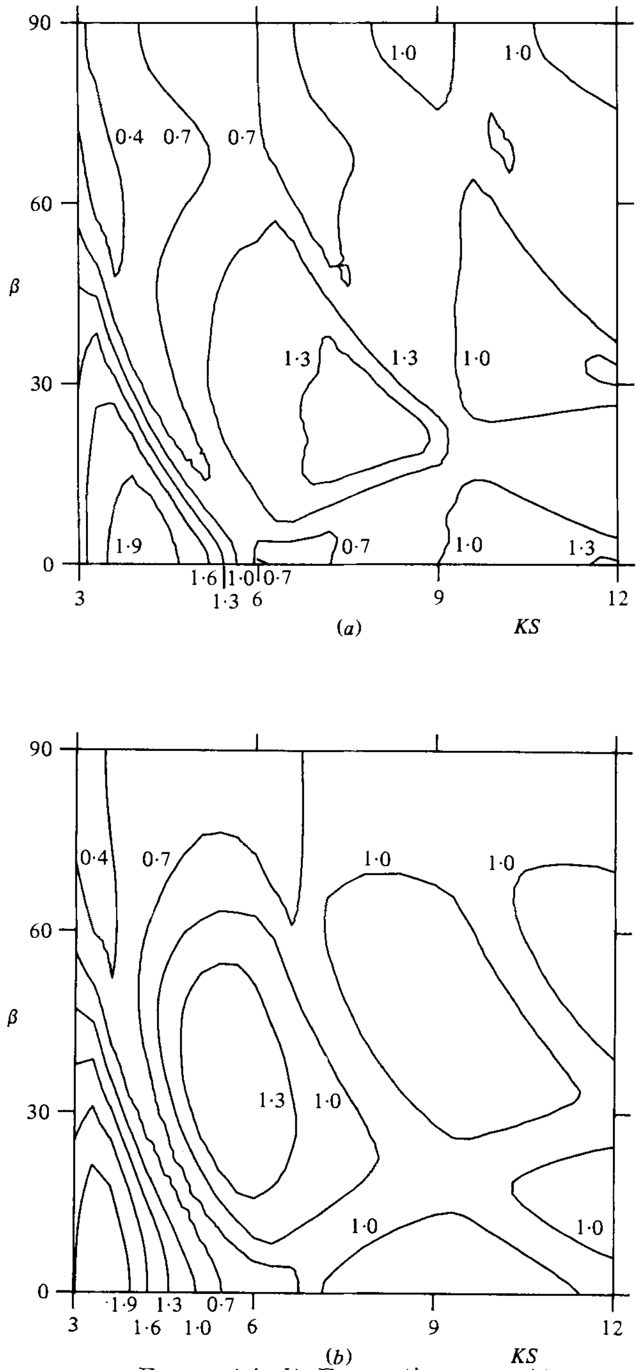

FIGURE 4 (*a*, *b*). For caption see p. 14.

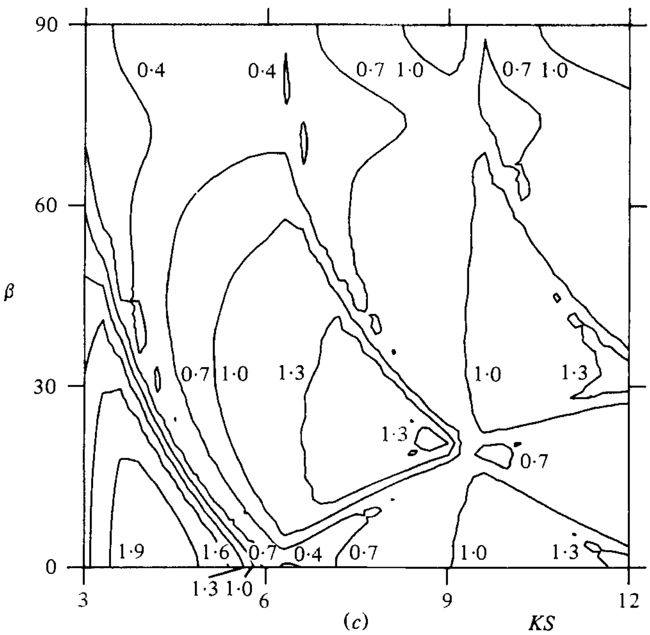

FIGURE 4. Contours of the averaged interaction factor $\bar{q}$ plotted against dimensionless spacing KS and angle of incidence β. Contour levels are marked at intervals of 0·3. The results are for the submerged cylindrical duct of Simon (1981), with a diameter of one third the spacing. (a) An array of 5 devices, including scattering. (b) An array of 5 devices, but without scattering effects. (c) An array of 9 devices, including scattering.

interesting to look first at the added-damping matrix coefficients computed using (26). Figure 6 (a) shows the ratio of the principal damping coefficient B_{11} (computed using the present method) to $|\hat{E}|^2 = D_r$, the damping coefficient for a single device. This ratio is compared to the constant value of unity predicted from the point-absorber theory. Results here are for an array of two devices, plotted against the dimensionless spacing KS. Two curves are shown, corresponding to different device radii. Clearly the larger devices (with greater scattering effect) alter the damping coefficient more. For comparison, figure 6 (b) shows results of Count & Jefferys (1980) obtained using a finite-element numerical technique for cylinders, with hemispherical bases, which intersect the free surface. The diameter of their device is roughly that of the smaller device of figure 6 (a). It is apparent that the deviations from unity are much greater in figure 6 (b), and this is possibly due to the much greater scattering of a surface-piercing device.

Figure 7 (a) shows the ratio of the off-diagonal damping coefficient B_{12} to D_r, plotted against spacing. The point-absorber theory of Evans (1980) would give this value as $J_0(KS)$, and the results from the present method display a 'shift' towards smaller spacing; this shift is more pronounced for the larger device. Figure 7 (b) shows the equivalent result from Count & Jefferys (1980), with the shift being towards *larger* spacing. This apparent contradiction between the two methods is another result of the fact that their devices intersect the free surface, since there is a phase lead ($\alpha < 0$) for such a device. This can be explained as follows.

When the number of devices is small the major contribution to an off-diagonal

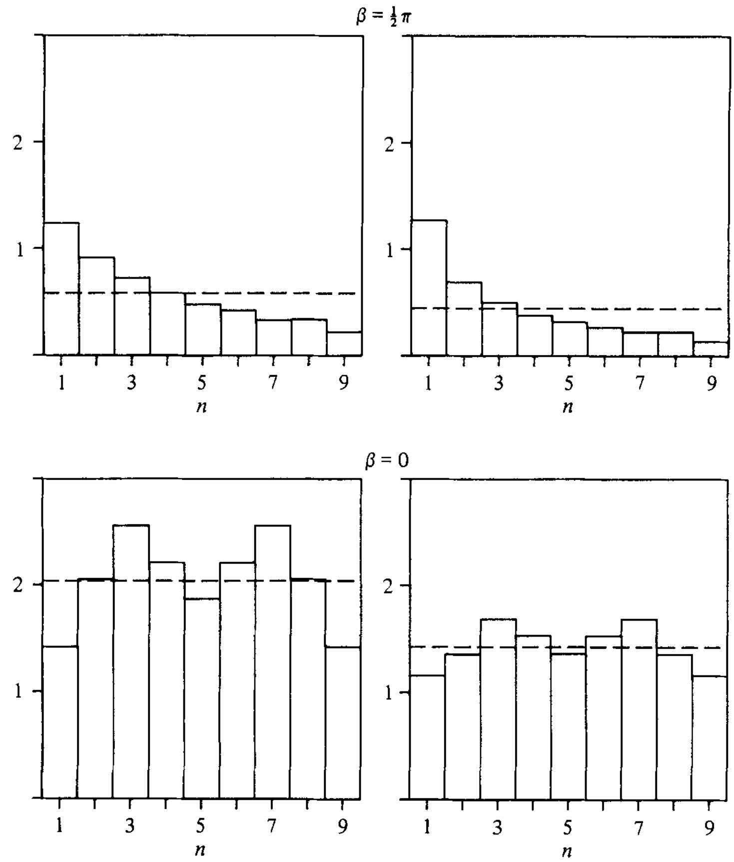

FIGURE 5. Histograms of the individual interaction factors for the devices for an array of 9 devices with a dimensionless spacing $KS = 4{\cdot}5$. Upper: head seas. Lower: beam seas. In each case the left-hand plots include scattering effects, the right-hand ones do not. Dashed lines: the average interaction factor for the array.

damping (or mass) coefficient is from a 'direct path' through the fluid; that is, waves radiated by one device affecting another without any scattering. Other paths through the fluid (for example scattering off a third device on the way from one device to the other) give a weaker contribution; it is only when there are more devices that these weaker contributions add up significantly. Thus the shift observed must be due to the finite size of the devices, not due to scattering. So it must be due to the phase lag (or lead) of the wave radiated by the first device, and also the phase lag (or lead) of the exciting force on the second device. Since the devices are identical, so are these phase effects, and this is the reason for the $e^{-2i\alpha}$ appearing in (35). Hence a good approximation for a few devices (and a first approximation in all cases) is to ignore scattering, in which case

$$c_p^{(q)} + d_p^{(q)} = H_0^{(2)}(KS|p-q|) \quad (p \neq q), \tag{40a}$$

and so

$$\begin{aligned} B_{pq} + iM_{pq} &= D_r e^{-2i\alpha} H_0^{(2)}(KS|p-q|) \\ &\approx D_r H_0^{(2)}(2\alpha + KS|p-q|). \end{aligned} \tag{40b}$$

This explains the difference in direction of the shift, since $\alpha > 0$ is general for submerged devices, and $\alpha < 0$ for floating ones (see e.g. Lighthill 1979).

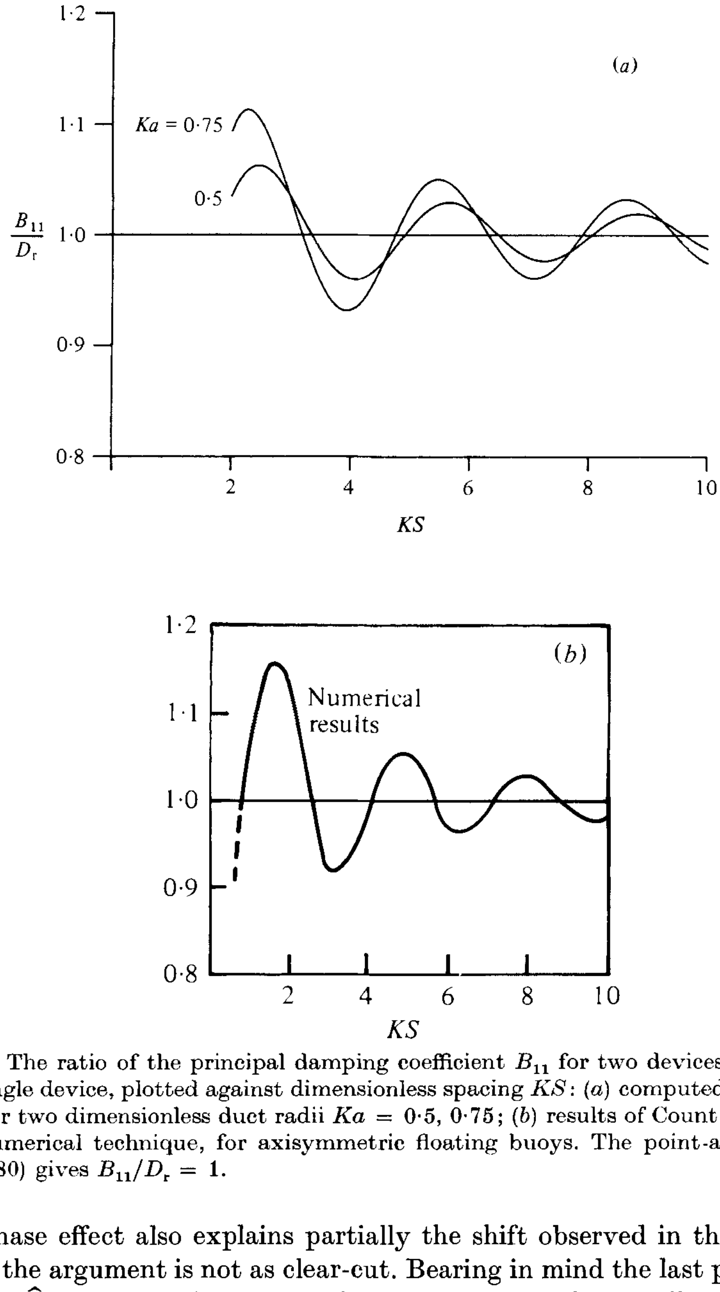

FIGURE 6. The ratio of the principal damping coefficient B_{11} for two devices to the coefficient D_r for a single device, plotted against dimensionless spacing KS: (a) computed using the present method, for two dimensionless duct radii $Ka = 0{\cdot}5, 0{\cdot}75$; (b) results of Count & Jefferys (1980), using a numerical technique, for axisymmetric floating buoys. The point-absorber theory of Evans (1980) gives $B_{11}/D_r = 1$.

This phase effect also explains partially the shift observed in the contours of $\bar{q}$, although the argument is not as clear-cut. Bearing in mind the last paragraph of § 6, neglecting $\hat{A}_n(n \neq 0)$, and concentrating on nearest-neighbour effects suggests that

$$P^{(j)}_{n,n\pm 1} = \hat{A}_0 H_0^{(2)}(KS) \left.\begin{array}{ll} \approx -\tfrac{1}{2}H_0^{(2)}(KS) & \text{including scattering} \\ \approx -\tfrac{1}{2}H_0^{(2)}(KS+2\alpha) & \text{no scattering,} \end{array}\right\} \quad (41)$$

which in turn suggests a shift of $2\alpha\,(\approx 0{\cdot}5\text{–}1{\cdot}0)$ towards higher values of KS, as observed.

Figures 8 (a, b) show diagonal and off-diagonal damping coefficients for 3, 5 and 9 devices. The matrix elements shown are near the centre of each array; that is $B_{\hat{N}\hat{N}}$ and $B_{\hat{N}\hat{N}+1}$, where $\hat{N} = \tfrac{1}{2}(N+1)$. Note the large deviations from the theory of Evans

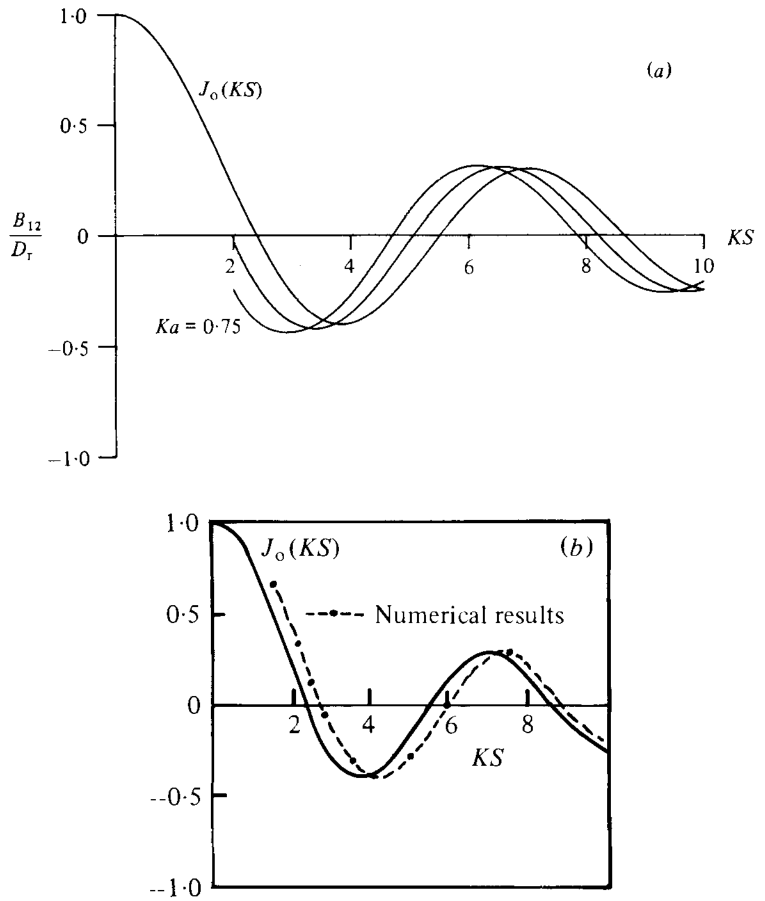

FIGURE 7. The ratio of the cross-coupling damping coefficient B_{12} for two devices to the coefficient D_r for a single device, plotted against dimensionless spacing KS: (*a*) computed using the present method, for two dimensionless duct radii. $Ka = 0{\cdot}5$, $0{\cdot}75$; (*b*) results of Count & Jefferys (1980), using a numerical technique, for axisymmetric floating buoys. The point-absorber theory of Evans (1980) gives $B_{12}/D_r = J_0(KS)$.

(1980) that occur at certain isolated frequencies when the number of devices is large. It is impossible to display all the important quantities, so the graphs displayed should be taken as representative of results for general values. There is in principle no limit to the number of devices in an array which this method can solve, and the limit imposed by computer time and storage is about 25 devices. However, the graphs become increasingly intricate, with a lot of fine structure, as the number of devices increases, and this does not add to one's understanding. So graphs shown will be for moderate numbers of devices, and, for comparison with Thomas & Evans (1981), $N = 5$ is of special interest.

Figures 9(*a*, *b*) show diagonal and off-diagonal added-mass coefficients for 2 and 5 devices, compared with the results for the limit of point absorbers. The matrix coefficients have been normalized to match the added length $\tilde{l}$ of Simon (1981, figure 11), with $\sigma = 0{\cdot}5$. Notice once more that the principal effect is a shift towards lower frequency, as explained before (40).

Figures 10(*a*, *b*) show the modulus and phase of the factor $\gamma_n(\beta)$ by which the

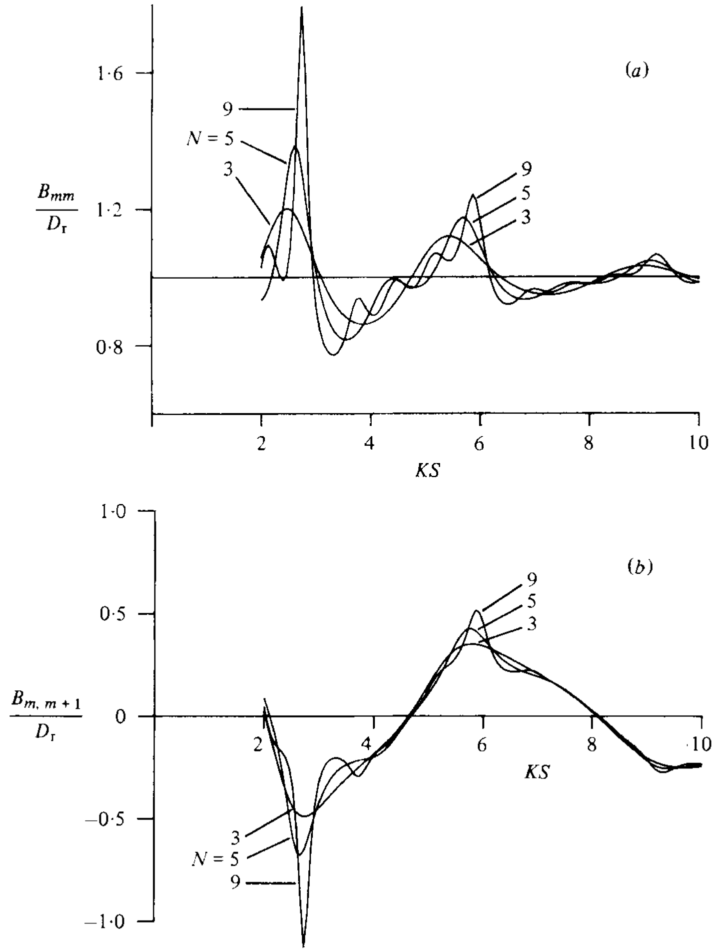

FIGURE 8. Diagonal and off-diagonal damping-coefficient ratios B_{ij}/D_r for 3, 5 and 9 devices, plotted against dimensionless spacing KS. Each device is a duct of dimensionless radius $\frac{1}{4}KS$. (a) The central coefficient B_{mm}/D_r, $m = \frac{1}{2}(N+1)$. (b) The off-diagonal coefficient $B_{m,m+1}/D_r$, $m = \frac{1}{2}(N+1)$.

exciting force is augmented; this is for an array of 5 devices in normally incident waves (beam seas, $\beta = 0$). Three curves are given for the first, second and third devices, and symmetry shows that the outer pair must be subject to the same force; similarly, the forces on the second and fourth devices must be the same. Notice that the force can be nearly twice as large as the value given by point-absorber theory, which is just the value for an isolated device; that the devices are subject to virtually the same forces in a band around $KS = 6$; that at $KS = 2\pi$ the phase of these forces is a minimum; and at the same value the forces are virtually unchanged in magnitude.

Figures 11 (a, b) show the corresponding results for waves incident along the direction of the array (head seas or $\beta = \frac{1}{2}\pi$). The three curves given are for the first, third and fifth devices in an array of five; no longer are the outer pair subject to the same forces when scattering is included, except in beam seas. Of note here are interesting effects which occur when the spacing is a multiple of half a wavelength, but more important is the force on the rear (fifth) device. This device feels an exciting force

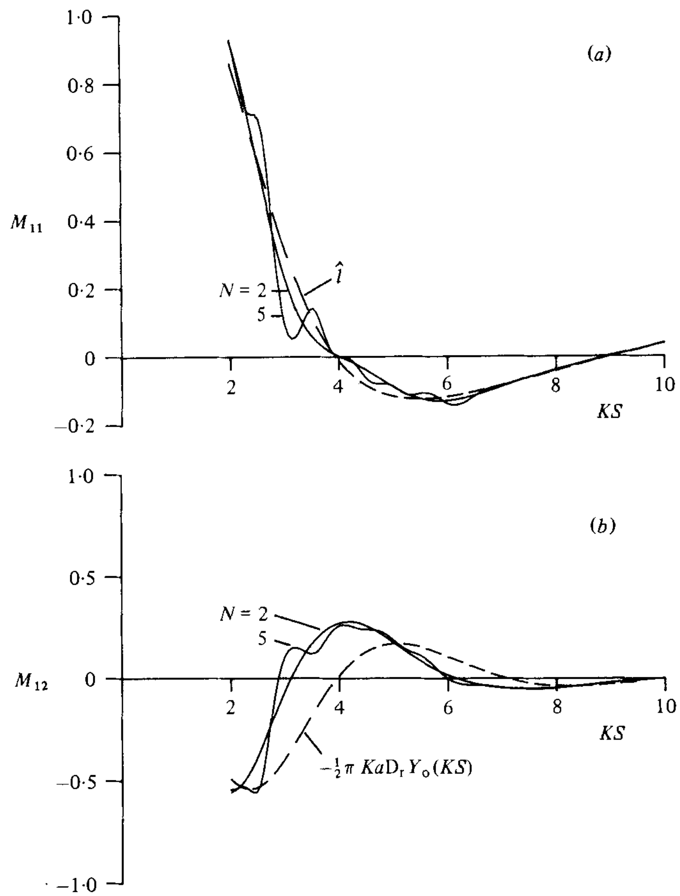

FIGURE 9. Diagonal and off-diagonal added-mass coefficients M_{ij}, normalized so that the diagonal elements reduce to the added length $\hat{l}$ of Simon (1981) as the spacing-to-duct-diameter ratio becomes large. Results are shown for 2 and 5 devices, and the point-absorber limit, plotted against the dimensionless spacing KS. Each device has a duct radius of $\frac{1}{4}S$: (*a*) M_{11} compared with the point-absorber value $\hat{l}$; (*b*) M_{12} compared with $-\frac{1}{2}\pi KaD_r Y_0(KS)$.

which is always *greater* than the force on an isolated device, and this force always *lags* behind the wave. This last result has been thought to account for the 'wave-shortening' effect observed as waves pass along an attenuator-type device such as the Lancaster flexible bag (D. P. Hurdle, private communication).

The foregoing graphs show that the damping matrix and the exciting force are both significantly modified when scattering is included. It would therefore be expected that the optimal q-factor, defined by (29), would also be significantly different from the point-absorber value given by Evans (1980). Figure 12 shows the unexpected result that there is remarkable agreement between the values with and without scattering. This conclusion is also reached by Count & Jefferys (1980) in their numerical study. This close agreement is as yet almost wholly unexplained. Nor is it yet known why Greenhow (1980, figure 18) finds that the q-factor *is* changed (shifted to higher frequency) for an array of two heaving spheres. It is hoped that further work on arrays of wave-energy devices, studying different geometries and using different methods, will explain this.

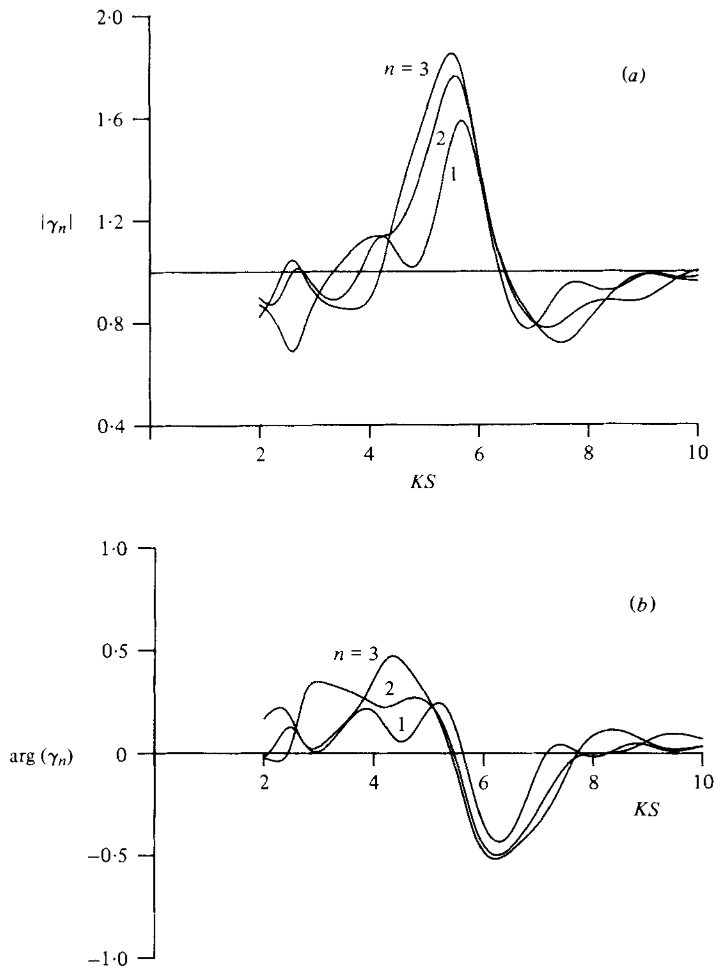

FIGURE 10. Modulus (*a*) and phase (*b*) of the factor $\gamma_n(\beta)$ which represents the augmentation of exciting forces due to scattering in the array. The point-absorber theory of Evans (1980) would give this factor to be identically unity. Curves are for an array of five devices, in beam seas ($\beta = 0$). Each device is of dimensionless radius $\frac{1}{4}KS$, where KS is the dimensionless spacing. The numbering denotes the appropriate device.

Finally, figures 13(*a*, *b*) show the optimal amplitudes of the device responses. (These quantities are actually the ratios between the displacement of the air/water interface inside the ducts to the wave height (see Simon 1981).) For comparison the optimal response of an isolated device is shown. The wave height has unit amplitude on this scale, and so it is seen that the device responses are not too large for linear theory to be valid. The exception to this is for $KS < 3$ (not shown), where the optimal responses become much larger, just as does the optimal response of a *single* device. Thus it is concluded that the enormous amplitudes displayed in Thomas & Evans (1981) are the result of the choice of device, rather than the array interactions; this can be seen by evaluating the optimal amplitude of a single heaving semi-immersed sphere. In their notation this is given by $(2\pi\kappa^2a^2\,\Omega)^{-1}$, which is about 3·2 when $\kappa a = 0{\cdot}4$. It would seem that the ratio of the optimal amplitude of a device in an array to its optimal amplitude in isolation is typically between 0·5 and 2 (beam seas), or 0·5 and 1

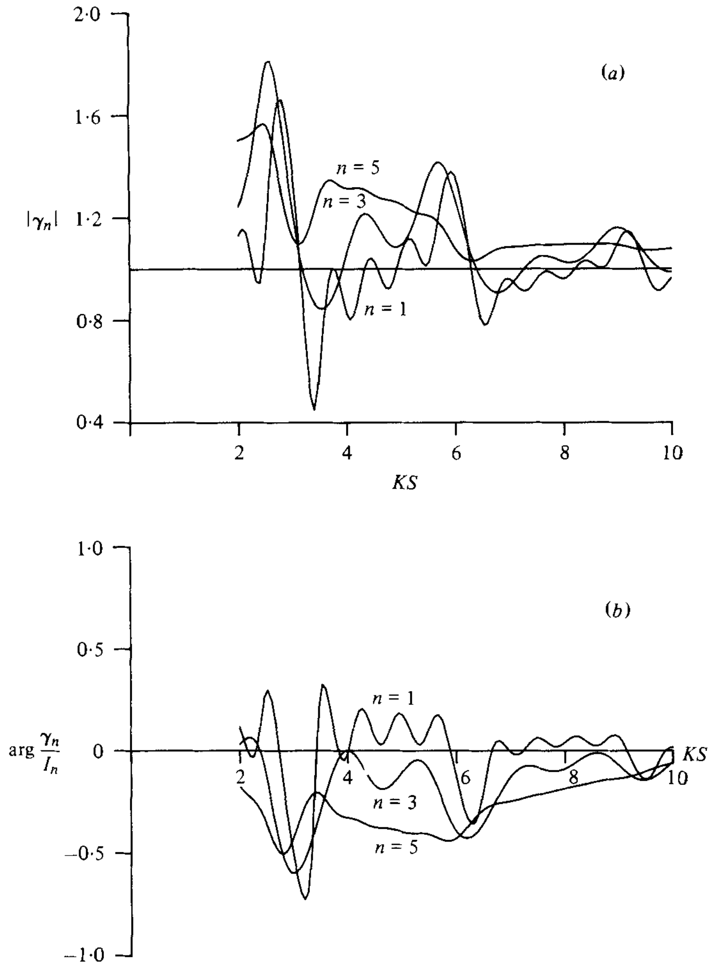

FIGURE 11. As for figure 10, for head seas ($\beta = \frac{1}{2}\pi$). Note that the phase of $\gamma_n(\frac{1}{2}\pi)$ is plotted relative to the incident wave.

(head seas), whether or not scattering is included. This simply re-emphasizes the need to optimize the properties of the individual devices.

As a summary of the results of this paper it may be said that important effects arise in arrays due to the relative *phases* of exciting forces and displacements, rather than their magnitudes, and so scattering and finite size can each act to produce substantial frequency shifts, through alteration of these phases.

8. Conclusion

A matrix method has been described that models the multiple-scattering problem in an array of wave-energy absorbers. It has been shown that scattering is important in the true (non-optimised) energy absorption of an array; that is, where the phases and amplitudes of the device responses are not controlled. The main effect of scattering is to *shift* the frequency response of the array. Similarly it is shown that the added-damping and added-mass matrices are substantially altered, with the main effect, when there are few devices, being a shift in the frequency response. This change from

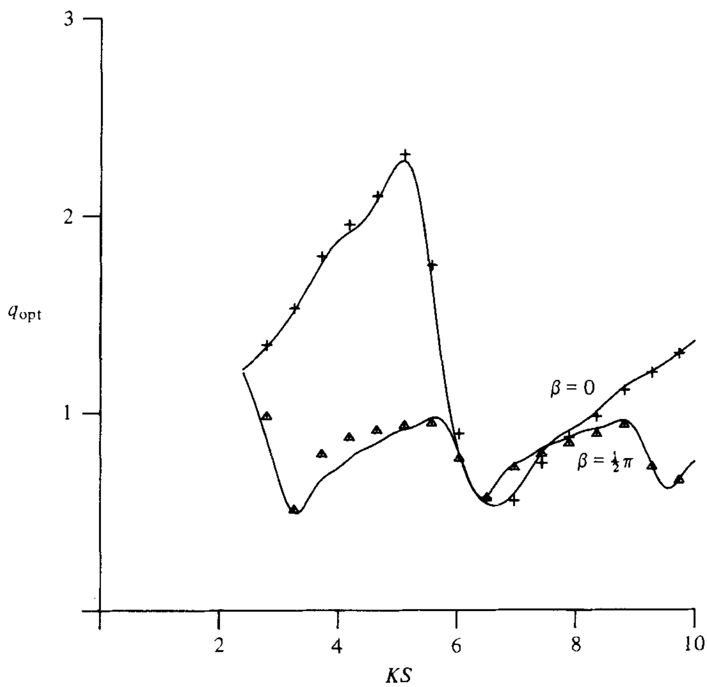

FIGURE 12. The optimal q-factor, for an array of five devices, plotted against dimensionless spacing KS. Each device has a radius of one quarter of the spacing. The gain factor q_{opt} represents the amount by which the array, under optimal conditions, improves upon the optimal performance of N isolated devices. Curves: the point-absorber theory of Evans (1980). Present method including scattering: $+$, $\beta = 0$; Δ, $\beta = \frac{1}{2}\pi$.

point-absorber theory is principally due to phase effects of the finite size of devices, rather than their scattering effect. It is also shown that the exciting forces on array elements are significantly modified in both phase and amplitude owing to the inclusion of scattering. Despite this, the maximum possible energy absorption of a controlled array is virtually unaffected by scattering among the devices.

Further work will investigate another matrix technique for the study of arrays, one which allows the study of horizontal modes of energy extraction. The case of an infinite array will also be studied, and results will be presented for other device geometries including the sphere.

I would like to thank Sir James Lighthill for all the help and encouragement he has given me during this work, and also Dr D. V. Evans and Mr B. M. Count for their advice and comments, and the Science Research Council for the financial support that made this research possible. This work was carried out while I was a research student at D.A.M.T.P., Cambridge.

Appendix. Errors due to the plane-wave approximation

Thus the replacement of the diverging-wave potential (1) by the plane-wave potential (5) results in errors that are in some sense small, and an attempt will now be made to show the order of the approximation involved, based on the restriction to axisymmetric bodies making vertical motions. With this restriction, the exciting force on device 2, due to the wave (1) from device 1, is given *exactly* by the exciting

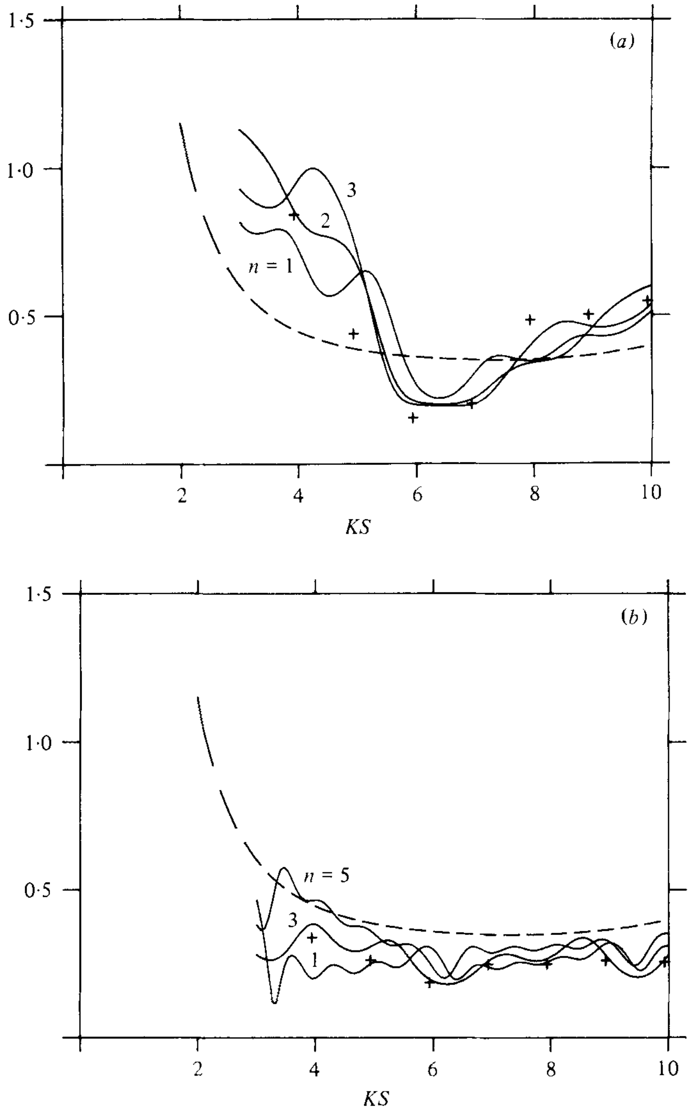

FIGURE 13. The device response needed to give the optimal q-factor of figure 12, plotted against dimensionless spacing KS: (a) beam seas ($\beta = 0$), showing the responses of the first, second and third devices; (b) head seas ($\beta = \frac{1}{2}\pi$), showing the responses of the first, third and fifth devices. In each case the dashed line shows the optimal response amplitude of an isolated device, and the '+' points are the optimal amplitude of the third device under the point-absorber approximation.

force due to wave (5) with amplitude given by (6). Wave (5) does however scatter differently from wave (1), and so the back effect on device 1 will be changed due to the approximation. The difference can be calculated as follows.

The diffraction potential due to wave (1), rewritten as (3), scattering off device 2 will be

$$\begin{aligned} &= e^{-Kz}\sum_m \Big[\sum_n a_n H_{n-m}\Big] A_m H_m^{(2)}(Kr_2)\, e^{im\theta_2}(-i)^m \\ &= e^{-Kz}\sum_m \Big[\sum_n a_n H_{n-m}\Big] A_m\Big[\sum_l H_{l-m}(-i)^l J_l(Kr_1)\, e^{il\theta_1}\Big] \\ &= e^{-Kz}\sum_l \Big[\sum_m\sum_n a_n A_m H_{n-m} H_{l-m}\Big](-i)^l J_l(Kr_1)\, e^{il\theta_2}. \end{aligned} \tag{A 1}$$

The scattering coefficients A_m are defined for the particular device in the same manner as (12*b*); H_m stands for $H_m^{(2)}(KS)$, and Graf's addition theorem has again been used, to re-express the potential in terms of (r_1, θ_1, z) co-ordinates. The exciting force due to the back effect is proportional to the $l = 0$ term, which is thus

$$\sum_m \sum_n a_n A_m H_{n-m} H_m(-1)^m. \tag{A 2}$$

Conversely, if the plane-wave approximation is used, it is the wave (5) that scatters off device 2 to yield a back effect, and so the corresponding expressions are

$$e^{-Kz} \sum_l [(\sum_n a_n H_n) \sum_m (-i)^m A_m H_{l-m}] (-i)^l J_l(Kr_1) e^{il\theta_1}, \tag{A 3}$$

$$\Sigma\Sigma\, a_n A_m H_n H_m(+i)^m. \tag{A 4}$$

The difference between (A 2) and (A 4), which represents the error in the exciting force due to the approximation, is

$$\sum_m \sum_n a_n A_m(-1)^m H_m\{H_{n-m} - (-i)^m H_n\}. \tag{A 5}$$

Notice that this is $O((KS)^{-2})$, whereas each of (A 2) and (A 4) is $O((KS)^{-1})$; further notice that (A 5) is identically zero if $A_m = 0$ for $m \neq 0$. The latter comment shows that the difference will be small compared with (A 2) if A_0 is sufficiently large in comparison to the other scattering coefficients A_n, and the former comment shows that the error is of the order of the *fourth* scattering. (If the analogy of the wave with a particle is allowed, then the error is introduced when the particle, coming from device 1, has bounced off device 2, device 1 and device 2 again, losing some 'momentum' at each bounce, and then goes back to effect device 1 once more.) Work by Ohkusu (1973), using an iterative technique, suggests that higher-order scattering is relatively unimportant in the total potential, so it is felt that overall the plane-wave approximation will give reasonable accuracy for sensible parameter values, with the resulting huge simplification in the multiple scattering problem. Specifically, the work will involve parameter ranges for wavenumber K, interspacing S and device size L such that $KS \geqslant O(2\pi)$ and $S/L = O(2\pi)$, so that $KL \geqslant O(1)$ and scattering is important.

REFERENCES

Abramowitz, M. & Stegun, I. 1964 *Handbook of Mathematical Functions*. National Bureau of Standards.

Budal, K. 1977 Theory for absorption of wave power by a system of interacting bodies. *J. Ship Res.* **21**, 248–253.

Count, B. M. & Jefferys, E. R. 1980 Wave power: the primary interface. In *Proc. 13th Symp. Naval hydrodynamics, Tokyo*, pp. 817–826.

Davis, A. M. J. 1976 A relation between the radiation and scattering of surface waves by axisymmetric bodies. *J. Fluid Mech.* **76**, 85–88.

Evans, D. V. 1980 Some theoretical aspects of three-dimensional wave-energy absorbers. In *Proc. 1st Symp. Wave-Energy Utilisation, Gothenburg*, 1979. Chalmers Tech. Univ.

Greenhow, M. J. L. 1980 The hydrodynamic interactions of spherical wave-power devices in surface waves. In *Power from sea waves* (ed. B. M. Count), pp. 287–343. Academic.

Lighthill, M. J. 1979 Two-dimensional analyses related to wave-energy extraction by submerged resonant ducts. *J. Fluid Mech.* **91**, 253–317.

OHKUSU, M. 1973 Wave action on groups of vertical circular cylinders. In *Selected Papers from J. Soc. Naval Architects of Japan*, vol. 11, pp. 37–50.

OHKUSU, M. 1974 Hydrodynamic forces on multiple cylinders in waves. In *Proc. Int. Symp. Dynamics of Marine Vehicles and Structures in Waves, London* paper no. 12, pp. 107–112. Inst. Mech. Engrs.

SIMON, M. J. 1981 Wave-energy extraction by a submerged cylindrical resonant duct. *J. Fluid Mech.* **104**, 159–187.

SROKOSZ, M. A. 1979*a* Some theoretical aspects of wave-power absorption. Ph.D. thesis. Univ. Bristol.

SROKOSZ, M. A. 1979*b* The submerged sphere as an absorber of wave power. *J. Fluid Mech.* **95**, 717–741.

SROKOSZ, M. A. & EVANS, D. V. 1979 A theory for wave-power absorption by two independently oscillating bodies. *J. Fluid Mech.* **90**, 337–362.

THOMAS, G. P. & EVANS, D. V. 1981 Arrays of three-dimensional wave-energy absorbers. *J. Fluid Mech.* **108**, 67–88.